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UNIQUENESS AND STABILITY OF THIN-WALLED CYLINDERS UNDER INTERNAL PRESSURE, TENSION AND TORQUE

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Abstract—Using the method of R. Hill, the uniqueness of deformation and stability of a thin-walled, rigid-plastic cylinder is examined under internal pressure, tension and torque.

INTRODUCTION

In this short paper, we examine the uniqueness of deformation and the stability of a thin-walled, rigid-plastic cylinder under internal pressure, tension and torque. It will be shown that the effect of torque, applied at the ends of the cylinder, could be either to increase or to decrease the axial load (or the pressure) carrying capacity of the cylinder depending upon the value of the strain-hardening index. This paper, is, in fact, an extension of an earlier investigation by Kumar and Ariaratnam[1].

FORMULATION AND SOLUTION

Consider a thin-walled cylinder with open-ends subjected to an axial tensile load T, internal pressure p and torque M. In the current state, the cylinder is assumed to have thickness t and mean radius R. For the uniform mode of deformation, the stress components are

$$\sigma_{rr} = 0, \sigma_{\theta\theta} = pR/t, \sigma_{zz} = T/2\pi Rt, \sigma_{\theta z} = \sigma_{z\theta} = \tau(\text{say}). \quad (1)$$

When the cylinder-ends are closed, $\sigma_{zz} = T/2\pi Rt + pR/2t$. This stress-distribution satisfies the equilibrium conditions and, since the stress is everywhere at the yield point, also satisfies the Mises Yield condition

$$\sigma_{\theta\theta}^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + \sigma_{zz}^2 + 6\sigma_{\theta z}^2 = 6k^2 \quad (2)$$

where k is the yield stress in simple shear. To find out if the subsequent incremental deformation is unique (and hence stable), we use Hill's uniqueness criterion which can be referred to in [1]. Using the same notations as in [1], the strain-rates $\lambda_{ij} \equiv \Delta\epsilon_{ij}$ must satisfy the following relations in view of (1):

$$\begin{aligned} &\lambda_{rr} + \lambda_{\theta\theta} + \lambda_{zz} = 0, \lambda_{r\theta} = 0, \lambda_{rz} = 0 \\ &\lambda_{\theta z}/\lambda_{rr} = -3\tau/(\sigma_{\theta\theta} + \sigma_{zz}) = -3\beta/(1+\alpha) \\ &\lambda_{rr}/\lambda_{\theta\theta} = -(\sigma_{\theta\theta} + \sigma_{zz})/(2\sigma_{\theta\theta} - \sigma_{zz}) = -(1+\alpha)/(2\alpha - 1) \text{ at } r = R \end{aligned} \quad (3)$$

where $\alpha = \sigma_{\theta\theta}/\sigma_{zz}$, $\beta = \tau/\sigma_{zz}$. When expressed in terms of the physical components velocity u, v, w in the r-, θ-, and z-directions respectively, the preceding relations (3) become

$$\begin{aligned} &\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} = 0, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0 \\ &\times\frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right)\Big/\frac{\partial u}{\partial r} = -3\beta/(1+\alpha), \frac{\partial u}{\partial r}\Big/\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right) = -\frac{1+\alpha}{2\alpha - 1} \text{ at } r = R. \end{aligned} \quad (4)$$

The outline of getting the exact solution of such types of partial differential equations is given in [2]. The general solution for the velocity field in the present case is found to be simply

$$u = -A[r + 3\alpha R^2/(\alpha - 2)r], \; v = 12A\beta rz/R(2-\alpha), \; W = 2Az \tag{5}$$

where A is an arbitrary constant. This velocity field is much restricted compared to the cases when any two of the three loads act at a time[1] or when only internal pressure is present[3]. But this is not a matter of surprise. While dealing with rigid-plastic solids with a smooth yield surface, a wider class of admissible field is available for plane strain than for a genuinely three-dimensional case.

As in [1], all the terms in the uniqueness condition are evaluated using (1) and (5), and the resulting expression is integrated about the mean radius R; a rearrangement of terms gives the following inequality for the critical hardening rate h:

$$h > \sigma_{zz}\frac{(3\alpha^3 - \alpha^2 - 2\alpha + 2 + 3\beta^2\alpha + \beta^2)}{3(\alpha^2 - \alpha + 1 + 3\beta^2)}. \tag{6}$$

The hardening parameter h can be expressed in terms of the "critical subtangent" $\bar{z}$ on the generalized stress ($\bar{\sigma}$)–strain($\bar{\epsilon}$) curve (e.g. [1]):

$$h = \frac{2}{3}\frac{d\bar{\sigma}}{d\bar{\epsilon}} = \frac{2}{3}\frac{\bar{\sigma}}{\bar{z}} = \frac{2}{\sqrt{3}}\frac{k}{\bar{z}}, \; \frac{1}{\bar{z}} = \frac{1}{\bar{\sigma}}\frac{d\bar{\sigma}}{d\bar{\epsilon}}. \tag{7}$$

Then, (6) becomes

$$\frac{1}{\bar{z}} > \frac{(3\alpha^3 - \alpha^2 - 2\alpha + 2 + 3\beta^2\alpha + 3\beta^2)}{2(\alpha^2 - \alpha + 1 + 3\beta^2)^{3/2}}. \tag{6a}$$

For $\beta = 0$, the eqn (6a) yields the value of $1/\bar{z}$ above which the uniqueness is certainly guaranteed when the open-end cylinder is subjected to pressure and tension. For $\alpha = 0$ (i.e. only tension) and $\alpha = \infty$ (i.e. only pressure), we recover the familiar results reported, for example, in [4].

It may be mentioned here that since only the uniform mode of deformation is the only admissible field in (5), the instability occurs when one or more loads reach the maximum value. Hence, the equality sign in (6) or (6a) corresponds to the critical conditions.

Now, to investigate the effect of small torque (twist) on the axial load or the internal pressure carrying capacity of the cylinder, we consider a material model of the Ramberg–Osgood type:

$$k = k_0\bar{\epsilon}^m, \qquad 0 < m < 1 \tag{8}$$

as described in [1]. Using the procedure as in [1], we calculate the values of the critical axial load (or critical internal pressure) with and without the torque acting at the cylinder-ends. The results are summarised below.

(i) *Tension and torque* ($\alpha = 0$)

$$h = \frac{2}{3}\sigma_{zz}(1 + 1.5\beta^2)/(1 + 3\beta^2)$$

$$\frac{\text{Critical axial load torque}}{\text{Critical axial load without torque}} = \frac{1}{(1 + 1.5\beta^2)^m(1 + 3\beta^2)^{(1-3m)/2}} \tag{9}$$

(ii) *Pressure and torque* (*Open-ends*: $\sigma_{zz} = 0$, $\beta/\alpha = \tau/\sigma_{\theta\theta}$)

$$h = \sigma_{\theta\theta}\frac{(1+\tau^2/\sigma_{\theta\theta}^2)}{(1+3\tau^2/\sigma_{\theta\theta}^2)}$$

$$\frac{\text{Critical pressure with torque}}{\text{Critical pressure without torque}} = \frac{1}{(1+\tau^2/\sigma_{\theta\theta}^2)^m(1+3\tau^2/\sigma_{\theta\theta}^2)^{(1-3m)/2}} \tag{10}$$

(iii) *Pressure and torque* (*closed-ends*: $\alpha = 2$)

$$h = \sigma_{\theta\theta}\frac{(1+2\tau^2/\sigma_{\theta\theta}^2)}{(1+4\tau^2/\sigma_{\theta\theta}^2)}$$

$$\frac{\text{Critical pressure with torque}}{\text{Critical pressure without torque}} = \frac{1}{(1+2\tau^2/\sigma_{\theta\theta}^2)^m(1+4\tau^2/\sigma_{\theta\theta}^2)^{(1-3m)/2}}. \tag{11}$$

It is clear from relations (9)–(11) that a small twist (torque) can *increase or decrease* the critical load (pressure) depending upon the value of the strain-hardening index "m". This result is different from that reported in [1] wherein the error was because of an unfortunate calculation mistake.

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